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On construction of multivariate wavelets with vanishing moments ☆

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Abstract

Wavelets with matrix dilation are studied. An explicit formula for masks providing vanishing moments is found. The class of interpolatory masks providing vanishing moments is also described. For an interpolatory mask, formulas for a dual mask which also provides vanishing moments of the same order and for wavelet masks are given explicitly. An example of construction of symmetric and antisymmetric wavelets for a concrete matrix dilation is presented.

Keywords: Wavelet system; Matrix dilation; Unitary Extension Principle; Interpolatory mask

1. Introduction

We discuss construction of compactly supported biorthogonal wavelets with a matrix dilation. For image compression and some other applications, it is very desirable to have wavelets possessing vanishing moment property. It is well known how to provide this property in the one-dimensional case with dyadic dilation: a generating mask m_0 should be represented in the form $m_0(x) = (1 + e^{2\pi i x})^k T(x)$ (see, e.g., [1]). Situation is essentially different in the multidimensional case. Zero properties of masks cannot be described by means of factorization because no Euclid algorithm for multivariate polynomials exists.

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It is known [2] how to describe vanishing moment property in terms of linear identities (so-called *sum rule*). The sum rule is appropriate to check vanishing moment property for a given mask. However, finding masks by means of the sum rule is possible only numerically. One has to solve a linear systems which can contain a large enough number of unknowns. An explicit formula for masks providing vanishing moments would be more preferable in many situations. Some other descriptions of masks satisfying the sum rule are known, in particular, in terms of zero-conditions [2] and in terms of containment in a quotient ideal [3]. All these characterizations give methods for construction of required masks but do not allow to find an explicit general form. The goal of this paper is to present such a general form.

Let $\mathbb{N}$ be the set of positive integers, $\mathbb{R}^d$ denotes the d-dimensional Euclidean space, $x=(x_1,\dots,x_d)$, $y=(y_1,\dots,y_d)$ are its elements (vectors), $(x,y)=x_1y_1+\dots+x_dy_d$, $|x|=\sqrt{(x,x)}$, $\mathbf{e}_j=(0,\dots,1,\dots,0)$ is the jth unit vector in $\mathbb{R}^d$, $\mathbf{0}=(0,\dots,0)\in\mathbb{R}^d$; $\mathbb{Z}^d$ is the integer lattice in $\mathbb{R}^d$. For $x,y\in\mathbb{R}^d$, we write $x>y$ if $x_j>y_j$, $j=1,\dots,d$; $\mathbb{Z}_+^d=\{x\in Z^d\colon\ x\geqslant\mathbf{0}\}$. If $\alpha,\beta\in\mathbb{Z}_+^d$, $a,b\in\mathbb{R}^d$, we set $\alpha!=\prod_{j=1}^d\alpha_j!$, $\binom{\alpha}{\beta}=\frac{\alpha!}{\beta!(\alpha-\beta)!}$, $a^b=\prod_{j=1}^d {a_j}^{b_j}$, $[\alpha]=\sum_{j=1}^d\alpha_j$, $D^\alpha f=\frac{\partial^{[\alpha]}f}{\partial^{\alpha_1}x_1\dots\partial^{\alpha_d}x_d}$; δ_{ab} denotes Kronecker delta.

Let M be a non-degenerate $d\times d$ integer matrix whose eigenvalues are bigger than 1 in module, M^* is the conjugate matrix to M, I_d denotes the unit $d\times d$ matrix. We say that numbers $k,n\in\mathbb{Z}^d$ are congruent modulo M (write $k\equiv n \pmod M$) if $k-n=M\ell$, $\ell\in\mathbb{Z}^d$. The integer lattice $\mathbb{Z}^d$ is splitted into cosets with respect to the introduced relation of congruence. The number of cosets is equal to $|\det M|$ (see, e.g., [5, p. 107]). Let us take an arbitrary representative from each coset, call them digits and denote the set of digits by $D(M)$. Throughout the paper we consider that such a matrix M is fixed, $m=|\det M|$, $D(M)=\{s_0,\dots,s_{m-1}\}$, $s_0=\mathbf{0}$, $r_k=M^{-1}s_k$, $k=1,\dots,m-1$, $R(M)=\{r_0,\dots,r_{m-1}\}$.

We will consider wavelets constructed in the framework of multiresolution analysis (see [5, Chapter 5], [7]). Let a MRA in $L_2(\mathbb{R}^d)$ be generated by a scaling function φ which satisfies the refinement equation

$$\hat\varphi(x)=m_0\bigl({M^*}^{-1}x\bigr)\hat\varphi\bigl({M^*}^{-1}x\bigr),$$

where $m_0\in L_2([0,1]^d)$ is its mask (refinable mask). For any $m_\nu\in L_2([0,1]^d)$, there exists a unique set of functions $\mu_{\nu k}\in L_2([0,1]^d)$, $k=0,\dots,m-1$ (polyphase representatives of m_ν), so that

$$m_\nu(x)=\frac{1}{\sqrt m}\sum_{k=0}^{m-1}e^{2\pi i(s_k,x)}\mu_{\nu k}(M^*x). \tag{1}$$

The functions $\mu_{\nu k}$ can be expressed by

$$\mu_{\nu k}(x)=\frac{1}{\sqrt m}\sum_{s\in D(M^*)}e^{-2\pi i(M^{-1}s_k,x+s)}m_\nu\bigl({M^*}^{-1}(x+s)\bigr).$$

It is clear form this formula that a function m_ν is differentiable (n times) on $R(M^*)$ if and only if its polyphase representatives $\mu_{\nu k}$, $k=0,\dots,m-1$, are differentiable (n times) at the origin. If m_ν is a trigonometric polynomial, then the polyphase representatives, are also a trigonometric polynomials.

Now, let another MRA be generated by a scaling function $\tilde\varphi$ with a mask $\tilde m_0$ such that the integer shifts of φ, $\tilde\varphi$ are biorthogonal. According to *Unitary Extension Principle* (see [5,6]), to construct biorthogonal wavelets we should find wavelet masks m_ν, $\tilde m_\nu$, $\nu=1,\dots,m-1$, so that the polyphase matrices

$$\mathcal{M}:=\{\mu_{\nu k}\}_{\nu,k=0}^{m-1},\qquad \widetilde{\mathcal{M}}:=\{\tilde\mu_{\nu k}\}_{\nu,k=0}^{m-1},$$

satisfy

$$\mathcal{M}\widetilde{\mathcal{M}}^* = I_m, \tag{2}$$

and define wavelet functions by

$$\hat{\psi}^{(\nu)}(x) = m_\nu\big(M^{*-1}x\big)\hat{\varphi}\big(M^{*-1}x\big),$$
$$\widehat{\tilde{\psi}^{(\nu)}}(x) = \tilde{m}_\nu\big(M^{*-1}x\big)\hat{\tilde{\varphi}}\big(M^{*-1}x\big).$$

The corresponding dual systems consisting of the functions $\psi_{jk}^{(\nu)} = \psi^{(\nu)}(M^j\cdot + k)$, $\tilde{\psi}_{jk}^{(\nu)} = \tilde{\psi}^{(\nu)}(M^j\cdot + k)$, $j, k \in \mathbb{Z}^d$, are biorthogonal.

Throughout the paper we will consider that wavelet systems $\{\psi_{jk}^{(\nu)}\}$, $\{\tilde{\psi}_{jk}^{(\nu)}\}$ are constructed by means of Unitary Extension Principle from generating scaling functions $\varphi, \tilde{\varphi}$ whose masks $m_0, \tilde{m}_0$ are continuous at the origin and $m_0(\mathbf{0}) = \tilde{m}_0(\mathbf{0}) = 1$.

2. Polyphase criterion for vanishing moments

It is well known that the order of vanishing moments is one of the most important factors for success of wavelets in various applications. In particular, vanishing moments are necessary for smoothness of wavelets (see [4, Th. 3.3]) and guarantee the approximation order (see [2, Th. 2.1]).

Definition 1. We say that a wavelet system $\{\psi_{jk}^{(\nu)}\}$ has vanishing moments up to order α, $\alpha \in \mathbb{Z}_+^d$ (has *VM_α property* in the sequel), if $D^\beta \hat{\psi}^{(\nu)}(\mathbf{0}) = 0$, $\nu = 1, \dots, m-1$, for all $\beta \in \mathbb{Z}_+^d$, $\beta \leqslant \alpha$.

Theorem 2. *Let $\alpha \in \mathbb{Z}_+^d$, the functions $\hat{\varphi}, \tilde{m}_0, m_1, \dots, m_{m-1}$ have continuous derivatives up to order α on the set $R(M^*)$. The following conditions are equivalent*:

(i) *VM_α property is valid for $\{\psi_{jk}^{(\nu)}\}$;*
(ii) *$D^\beta(m_\nu(M^{*-1}x))|_{x=\mathbf{0}} = 0$, $\nu = 1, \dots, m-1$, for all $\beta \in \mathbb{Z}_+^d$, $\beta \leqslant \alpha$;*
(iii) *there exist complex numbers λ_γ, $\gamma \in \mathbb{Z}_+^d$, $\gamma \leqslant \alpha$, such that $\lambda_{\mathbf{0}} = 1$,*

$$D^\beta \tilde{\mu}_{0k}(\mathbf{0}) = \frac{1}{\sqrt{m}} \sum_{\mathbf{0} \leqslant \gamma \leqslant \beta} \lambda_\gamma \binom{\beta}{\gamma} (-2\pi i r_k)^{\beta-\gamma} \tag{3}$$

for all $\beta \in \mathbb{Z}_+^d$, $\beta \leqslant \alpha$.

We will prove this theorem after the following statement.

Proposition 3. *If $\tilde{m}_0$ is so that condition* (iii) *of Theorem* 2 *is valid, then*

$$\lambda_\beta = D^\beta\big(\tilde{m}_0\big(M^{*-1}x\big)\big)\big|_{x=\mathbf{0}} \tag{4}$$

for all $\beta \in \mathbb{Z}_+^d$, $\beta \leqslant \alpha$.

Proof. By Leibniz formula and (3),

$$
\begin{aligned}
D^\beta\big(e^{2\pi i(r_k,x)}\tilde\mu_{0k}(x)\big)\big|_{x=\mathbf{0}} &= \sum_{\mathbf{0}\leqslant\gamma\leqslant\beta}\binom{\beta}{\gamma}D^\gamma\big(e^{2\pi i(r_k,x)}\big)\big|_{x=\mathbf{0}}D^{\beta-\gamma}\tilde\mu_{0k}(\mathbf{0})\\
&= \sum_{\mathbf{0}\leqslant\gamma\leqslant\beta}\binom{\beta}{\gamma}(2\pi i r_k)^\gamma D^{\beta-\gamma}\tilde\mu_{0k}(\mathbf{0})\\
&= \frac{1}{\sqrt m}\sum_{\mathbf{0}\leqslant\gamma\leqslant\beta}\binom{\beta}{\gamma}(2\pi i r_k)^\gamma\sum_{\mathbf{0}\leqslant\varepsilon\leqslant\beta-\gamma}\lambda_\varepsilon\binom{\beta-\gamma}{\varepsilon}(-2\pi i r_k)^{\beta-\gamma-\varepsilon}\\
&= \frac{1}{\sqrt m}\sum_{\mathbf{0}\leqslant\gamma\leqslant\beta}\sum_{\mathbf{0}\leqslant\varepsilon\leqslant\beta-\gamma}\lambda_\varepsilon\binom{\beta-\gamma}{\varepsilon}\binom{\beta}{\gamma}(-2\pi i r_k)^{\beta-\varepsilon}\prod_{j=1}^d(-1)^{-\gamma_j}\\
&= \frac{1}{\sqrt m}\sum_{\mathbf{0}\leqslant\varepsilon\leqslant\beta}\lambda_\varepsilon(-2\pi i r_k)^{\beta-\varepsilon}\binom{\beta}{\varepsilon}\sum_{\mathbf{0}\leqslant\gamma\leqslant\beta-\varepsilon}\binom{\beta-\varepsilon}{\gamma}\prod_{j=1}^d(-1)^{-\gamma_j}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\sum_{\mathbf{0}\leqslant\gamma\leqslant\beta-\varepsilon}\binom{\beta-\varepsilon}{\gamma}\prod_{j=1}^d(-1)^{-\gamma_j} &= \prod_{j=1}^d\sum_{0\leqslant\gamma_j\leqslant\beta_j-\varepsilon_j}\binom{\beta_j-\varepsilon_j}{\gamma_j}(-1)^{-\gamma_j}\\
&= \prod_{j=1}^d(1-1)^{\beta_j-\varepsilon_j} = \begin{cases}0, & \beta\neq\varepsilon,\\ 1, & \beta=\varepsilon,\end{cases}
\end{aligned}
$$

we have

$$
D^\beta\big(e^{2\pi i(r_k,x)}\tilde\mu_{0k}(x)\big)\big|_{x=\mathbf{0}} = \frac{\lambda_\beta}{\sqrt m},\quad k=0,\dots,m-1.
$$

It follows from (1) that

$$
D^\beta\big(\tilde m_0\big(M^{*-1}x\big)\big)\big|_{x=\mathbf{0}} = \frac{1}{\sqrt m}\sum_{k=0}^{m-1}D^\beta\big(e^{2\pi i(r_k,x)}\tilde\mu_{0k}(x)\big)\big|_{x=\mathbf{0}} = \lambda_\beta. \qquad \square
$$

So we established that given α, the set of parameters λ_β, $\beta\in\mathbb{Z}_+^d$, $\beta\leqslant\alpha$, in (3) is unique, and λ_β does not depend on α.

Proof of Theorem 2. By Leibniz formula,

$$
D^\alpha\hat\psi^{(\nu)}(\mathbf{0}) = \sum_{\mathbf{0}\leqslant\beta\leqslant\alpha}\binom{\alpha}{\beta}D^\beta m_\nu\big(M^{*-1}x\big)\big|_{x=0}D^{\alpha-\beta}\hat\varphi\big(M^{*-1}x\big)\big|_{x=0}.
$$

Since $\hat\varphi(\mathbf{0})=1$, this yields (i) $\iff$ (ii).

We will prove (ii) $\Rightarrow$ (iii) by induction on α. Check the initial step for $\alpha=0$. Assume that $m_\nu(\mathbf{0})=0$, $\nu=1,\dots,m-1$. It follows form (1) that

$$
\sum_{k=0}^{m-1}\mu_{\nu k}(\mathbf{0})=0,\quad \nu=1,\dots,m-1. \tag{5}
$$

On the other hand, by (2),

$$\sum_{k=0}^{m-1} \overline{\tilde{\mu}_{0k}(\mathbf{0})}\mu_{\nu k}(\mathbf{0}) = 0, \quad \nu = 1, \ldots, m-1.$$

Because of linear independence of the vectors $(\mu_{\nu 0}(\mathbf{0}), \ldots, \mu_{\nu,m-1}(\mathbf{0})) \in \mathbb{R}^m$, $\nu = 1, \ldots, m-1$, there exists λ so that

$$\tilde{\mu}_{00}(\mathbf{0}) = \cdots = \tilde{\mu}_{0,m-1}(\mathbf{0}) = \lambda.$$

Taking into account the condition $\tilde{m}_0(\mathbf{0}) = 1$ which is equivalent to

$$\frac{1}{\sqrt{m}}\big(\tilde{\mu}_{00}(\mathbf{0}) + \cdots + \tilde{\mu}_{0,m-1}(\mathbf{0})\big) = 1,$$

we obtain $\lambda = \frac{1}{\sqrt{m}}$.

For the inductive step we assume that (ii) is valid for $\alpha > \mathbf{0}$ and (iii) holds for all $\alpha' \in \mathbb{Z}_+^d$, $\alpha' < \alpha$. So, due to Proposition 3, there exist constants λ_γ, $\gamma \in \mathbb{Z}_+^d$, $\gamma < \alpha$ such that (3) holds for all $\beta < \alpha$. If $\gamma \in \mathbb{Z}_+^d$, $\gamma < \alpha$, due to (1) and Leibniz formula, we have

$$\frac{1}{\sqrt{m}} \sum_{\mathbf{0} \leqslant \beta \leqslant \alpha - \gamma} \binom{\alpha-\gamma}{\beta} \sum_{k=0}^{m-1} (2\pi i r_k)^{\alpha-\beta-\gamma} D^\beta \mu_{\nu k}(\mathbf{0}) = D^{\alpha-\gamma} m_\nu\big(M^{*-1}x\big)\big|_{x=\mathbf{0}} = 0. \tag{6}$$

It follows from (2) that

$$\sum_{k=0}^{m-1} \overline{\tilde{\mu}_{0k}}\mu_{\nu k} = 0, \quad \nu = 1, \ldots, m-1.$$

Differentiating this equality α times gives

$$\sum_{\mathbf{0} \leqslant \beta \leqslant \alpha} \binom{\alpha}{\beta} \sum_{k=0}^{m-1} \overline{D^{\alpha-\beta}\tilde{\mu}_{0k}(\mathbf{0})} D^\beta \mu_{\nu k}(\mathbf{0}) = 0. \tag{7}$$

Multiply (6) by $\binom{\alpha}{\alpha-\gamma}\overline{\lambda_\gamma}$ and subtract from (7). After the same manipulation with each $\gamma \in \mathbb{Z}_+^d$, $\gamma < \alpha$, we obtain

$$\begin{aligned}
0 &= \sum_{\mathbf{0} \leqslant \beta \leqslant \alpha} \binom{\alpha}{\beta} \sum_{k=0}^{m-1} \overline{D^{\alpha-\beta}\tilde{\mu}_{0k}(\mathbf{0})} D^\beta \mu_{\nu k}(\mathbf{0}) \\
&\quad - \frac{1}{\sqrt{m}} \sum_{\mathbf{0} \leqslant \gamma < \alpha} \binom{\alpha}{\alpha-\gamma} \overline{\lambda_\gamma} \sum_{\mathbf{0} \leqslant \beta \leqslant \alpha-\gamma} \binom{\alpha-\gamma}{\beta} \sum_{k=0}^{m-1} (2\pi i r_k)^{\alpha-\beta-\gamma} D^\beta \mu_{\nu k}(\mathbf{0}) \\
&= \sum_{\mathbf{0} \leqslant \beta \leqslant \alpha} \binom{\alpha}{\beta} \sum_{k=0}^{m-1} \Bigg(\overline{D^{\alpha-\beta}\tilde{\mu}_{0k}(\mathbf{0})} \\
&\qquad - \frac{1}{\sqrt{m}} \sum_{\substack{\gamma \neq \alpha \\ \mathbf{0} \leqslant \gamma < \alpha-\beta}} \overline{\binom{\alpha-\gamma}{\beta}\binom{\alpha}{\alpha-\gamma}\binom{\alpha}{\beta}^{-1} \lambda_\gamma (-2\pi i r_k)^{\alpha-\beta-\gamma}} \Bigg) D^\beta \mu_{\nu k}(\mathbf{0}).
\end{aligned}$$

From this, taking into account that

$$\binom{\alpha-\gamma}{\beta}\binom{\alpha}{\alpha-\gamma}\binom{\alpha}{\beta}^{-1}=\frac{(\alpha-\beta)!}{\gamma!(\alpha-\beta-\gamma)!}=\binom{\alpha-\beta}{\gamma}, \tag{8}$$

and using the inductive hypotheses, we have

$$\begin{aligned}0&=\sum_{\mathbf{0}\leqslant\beta\leqslant\alpha}\binom{\alpha}{\beta}\sum_{k=0}^{m-1}\Biggl(\overline{D^{\alpha-\beta}\tilde{\mu}_{0k}(\mathbf{0})}-\frac{1}{\sqrt{m}}\sum_{\substack{\gamma\neq\alpha\\ \mathbf{0}\leqslant\gamma\leqslant\alpha-\beta}}\overline{\binom{\alpha-\beta}{\gamma}\lambda_\gamma(-2\pi i r_k)^{\alpha-\beta-\gamma}}\Biggr)D^\beta\mu_{\nu k}(\mathbf{0})\\ &=\sum_{k=0}^{m-1}\Biggl(\overline{D^\alpha\tilde{\mu}_{0k}(\mathbf{0})-\frac{1}{\sqrt{m}}\sum_{\mathbf{0}\leqslant\gamma<\alpha}\binom{\alpha}{\gamma}\lambda_\gamma(-2\pi i r_k)^{\alpha-\gamma}}\Biggr)\mu_{\nu k}(\mathbf{0}).\end{aligned}$$

Similarly to the arguments for the initial step, it follows from (5) that there exists λ_α such that

$$D^\alpha\tilde{\mu}_{0k}(\mathbf{0})-\frac{1}{\sqrt{m}}\sum_{\mathbf{0}\leqslant\gamma<\alpha}\binom{\alpha}{\gamma}\lambda_\gamma(-2\pi i r_k)^{\alpha-\gamma}=\frac{\lambda_\alpha}{\sqrt{m}}.$$

Thus, (3) is valid for $\beta=\alpha$ as was to be proved.

The implication (iii) $\Rightarrow$ (ii) will be also proved by induction on α. If (3) is valid for $\alpha=\mathbf{0}$, then $\tilde{\mu}_{0k}(\mathbf{0})=1/\sqrt{m}$, $k=0,\dots,m-1$. It follows from (2) that

$$\mu_{\nu0}(\mathbf{0})+\cdots+\mu_{\nu,m-1}(\mathbf{0})=0,\quad \nu=1,\dots,m-1.$$

Hence, on the basis of (1), $m_\nu(\mathbf{0})=0$, $\nu=1,\dots,m-1$, what proves the initial step.

For the inductive step, we assume that (iii) is valid for $\alpha>\mathbf{0}$ and (ii) holds for all $\alpha'\in\mathbb{Z}_+^d$, $\alpha'<\alpha$, i.e.

$$D^{\alpha-\gamma}m_\nu\bigl(M^{*-1}x\bigr)\big|_{x=\mathbf{0}}=0,\quad \gamma\in\mathbb{Z}_+^d,\ \gamma\neq\mathbf{0},\ \gamma\leqslant\alpha.$$

This yields (6) for $\gamma\neq\mathbf{0}$. Multiply (6) by $\binom{\alpha}{\alpha-\gamma}\overline{\lambda_\gamma}$ and add to (1) differentiated α times. After the same manipulation with each $\gamma\in\mathbb{Z}_+^d$, $\gamma<\alpha$, we obtain

$$\begin{aligned}D^\alpha m_\nu\bigl(M^{*-1}x\bigr)\big|_{x=\mathbf{0}}&=\frac{1}{\sqrt{m}}\sum_{\mathbf{0}\leqslant\beta\leqslant\alpha}\binom{\alpha}{\beta}\sum_{k=0}^{m-1}(2\pi i r_k)^{\alpha-\beta}D^\beta\mu_{\nu k}(\mathbf{0})+\frac{1}{\sqrt{m}}\sum_{\mathbf{0}<\gamma\leqslant\alpha}\binom{\alpha}{\alpha-\gamma}\overline{\lambda_\gamma}\\ &\quad\times\sum_{\mathbf{0}\leqslant\beta\leqslant\alpha-\gamma}\binom{\alpha-\gamma}{\beta}\sum_{k=0}^{m-1}(2\pi i r_k)^{\alpha-\beta-\gamma}D^\beta\mu_{\nu k}(\mathbf{0})\\ &=\frac{1}{\sqrt{m}}\sum_{\mathbf{0}\leqslant\beta\leqslant\alpha}\binom{\alpha}{\beta}\sum_{\mathbf{0}\leqslant\gamma\leqslant\alpha-\beta}\overline{\lambda_\gamma}\binom{\alpha}{\alpha-\gamma}\binom{\alpha-\gamma}{\beta}\binom{\alpha}{\beta}^{-1}\\ &\quad\times\sum_{k=0}^{m-1}(2\pi i r_k)^{\alpha-\beta-\gamma}D^\beta\mu_{\nu k}(\mathbf{0}).\end{aligned}$$

Due to (8) and (3), this yields

$$
\begin{aligned}
D^{\alpha} m_{\nu}\big(M^{*-1}x\big)\big|_{x=\mathbf{0}} &= \frac{1}{\sqrt{m}} \sum_{\mathbf{0}\leqslant\beta\leqslant\alpha} \binom{\alpha}{\beta} \sum_{k=0}^{m-1} \overline{\sum_{\mathbf{0}\leqslant\gamma\leqslant\alpha-\beta} \lambda_{\gamma} \binom{\alpha-\beta}{\gamma} (-2\pi i r_k)^{\alpha-\beta-\gamma}} D^{\beta}\mu_{\nu k}(\mathbf{0}) \\
&= \sum_{\mathbf{0}\leqslant\beta\leqslant\alpha} \binom{\alpha}{\beta} \sum_{k=0}^{m-1} \overline{D^{\alpha-\beta}\tilde{\mu}_{0k}(\mathbf{0})} D^{\beta}\mu_{\nu k}(\mathbf{0}) = D^{\alpha}\left(\sum_{k=0}^{m-1} \overline{\tilde{\mu}_{0k}(x)}\mu_{\nu k}(x)\right)\Bigg|_{x=\mathbf{0}}.
\end{aligned}
$$

It follows from (2) that $D^{\alpha} m_{\nu}(M^{*-1}x)|_{x=\mathbf{0}} = 0$ as was to be proved. □

Usually it is more useful to control univariate order of vanishing moment property (for example, to apply Taylor formula).

Definition 4. We say that a wavelet system $\{\psi_{jk}^{(\nu)}\}$ has vanishing moments up to order n, $n \in \mathbb{Z}_+$ (has VM^n *property* in the sequel) if $D^{\beta}\hat{\psi}^{(\nu)}(\mathbf{0}) = 0$, $\nu = 1, \dots, m-1$, for all $\beta \in \mathbb{Z}_+^d$, $[\beta] \leqslant n$.

Theorem 5. *Let $n \in \mathbb{Z}_+$, $\hat{\varphi}, \tilde{m}_0, m_1, \dots, m_{m-1}$ have continuous derivatives up to order n on the set $R(M^*)$. The following conditions are equivalent*:

(i) *VM^n property is valid for $\{\psi_{jk}^{(\nu)}\}$*;
(ii) *$D^{\beta}(m_{\nu}(M^{*-1}x))|_{x=\mathbf{0}} = 0$, $\nu = 1, \dots, m-1$, for all $\beta \in \mathbb{Z}_+^d$, $[\beta] \leqslant n$*;
(iii) *there exist complex numbers λ_{γ}, $\gamma \in \mathbb{Z}_+^d$, $[\gamma] \leqslant n$, such that $\lambda_{\mathbf{0}} = 1$,*

$$
D^{\beta}\tilde{\mu}_{0k}(\mathbf{0}) = \frac{1}{\sqrt{m}} \sum_{\mathbf{0}\leqslant\gamma\leqslant\beta} \lambda_{\gamma} \binom{\beta}{\gamma} (-2\pi i r_k)^{\beta-\gamma} \tag{9}
$$

for all $\beta \in \mathbb{Z}_+^d$, $[\beta] \leqslant n$.

Proof of this theorem follows immediately from Theorem 2 and Proposition 3.

3. General forms for masks providing VM_{α} and VM^n

Theorems 2 and 5 allow to find concrete masks $\tilde{m}$ providing VM_{α} and VM^n properties for $\{\psi_{jk}^{(\nu)}\}$. Given sets of parameters

$$
\begin{aligned}
\Lambda_{\alpha} &:= \big\{\lambda_{\gamma}, \gamma \in \mathbb{Z}_+^d, \gamma \leqslant \alpha, \lambda_{\mathbf{0}} = 1\big\}, \quad \alpha \in \mathbb{Z}_+^d, \\
\Lambda^{n} &:= \big\{\lambda_{\gamma}, \gamma \in \mathbb{Z}_+^d, [\gamma] \leqslant n, \lambda_{\mathbf{0}} = 1\big\}, \quad n \in \mathbb{Z}_+,
\end{aligned}
$$

we put

$$
\tilde{m}^*(x, \Lambda_{\alpha}) := \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi i (s_k, x)} \sum_{\mathbf{0}\leqslant\beta\leqslant\alpha} g_{\beta}(M^* x) \sum_{\mathbf{0}\leqslant\gamma\leqslant\beta} \binom{\beta}{\gamma} \lambda_{\gamma} (-2\pi i r_k)^{\beta-\gamma}, \tag{10}
$$

$$
\tilde{m}^*\big(x, \Lambda^{n}\big) := \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi i (s_k, x)} \sum_{0\leqslant[\beta]\leqslant n} g_{\beta}(M^* x) \sum_{\mathbf{0}\leqslant\gamma\leqslant\beta} \binom{\beta}{\gamma} \lambda_{\gamma} (-2\pi i r_k)^{\beta-\gamma}, \tag{11}
$$

where g_β is a trigonometric polynomial such that $D^\gamma g_\beta(\mathbf{0}) = \delta_{\beta\gamma}$ for all $\gamma \in \mathbb{Z}_+^d$, $\gamma \ne \beta$, $\mathbf{0} \leqslant \gamma \leqslant \alpha$ ($0 \leqslant [\gamma] \leqslant n$ for (11)). It is easy to check that $\tilde m_0^*(\cdot, \Lambda_\alpha)$ and $\tilde m_0^*(\cdot, \Lambda^n)$ satisfy condition (iii) respectively of Theorem 2 and of Theorem 5. Functions g_β can be found, for example, by

$$
\begin{aligned}
&g_\beta(x) = \prod_{j=1}^d g_{\beta_j}(x_j),\\
&g_{\beta_j}(u) = \frac{1}{\beta_j!(-2\pi i)^{\beta_j}}\left(\left(1 - e^{2\pi i u}\right)^{\beta_j} - \sum_{l=1}^{\alpha_j - 1} a_l\left(1 - e^{2\pi i u}\right)^{\beta_j + l}\right), \qquad (12)\\
&a_l = \frac{(-2\pi i)^{-\beta_j - l}}{(\beta_j + l)!}\frac{d^{\beta_j + l}}{du^{\beta_j + l}}\left(\left(1 - e^{2\pi i u}\right)^{\beta_j} - \sum_{r=1}^{l-1} a_r\left(1 - e^{2\pi i u}\right)^{\beta_j + r}\right)\Bigg|_{u=0}
\end{aligned}
$$

($\alpha_j = n$ in (12) for (11)). By this formula we have

$$
\begin{aligned}
\alpha_j = 1:\quad & g_1(u) = -\frac{1}{2\pi i}\left(1 - e^{2\pi i u}\right),\\
\alpha_j = 2:\quad & g_1(u) = -\frac{1}{2\pi i}\left(\left(1 - e^{2\pi i u}\right) + \frac{1}{2}\left(1 - e^{2\pi i u}\right)^2\right),\\
& g_2(u) = -\frac{1}{8\pi^2}\left(\left(1 - e^{2\pi i u}\right)^2 + \left(1 - e^{2\pi i u}\right)^3\right),\\
\alpha_j = 3:\quad & g_1(u) = -\frac{1}{2\pi i}\left(\left(1 - e^{2\pi i u}\right) + \frac{1}{2}\left(1 - e^{2\pi i u}\right)^2 + \frac{1}{3}\left(1 - e^{2\pi i u}\right)^3\right),\\
& g_2(u) = -\frac{1}{8\pi^2}\left(\left(1 - e^{2\pi i u}\right)^2 + \left(1 - e^{2\pi i u}\right)^3 + \frac{11}{12}\left(1 - e^{2\pi i u}\right)^4\right),\\
& g_3(u) = \frac{1}{48\pi^3 i}\left(\left(1 - e^{2\pi i u}\right)^3 + \frac{3}{2}\left(1 - e^{2\pi i u}\right)^4 + \frac{7}{4}\left(1 - e^{2\pi i u}\right)^5\right).
\end{aligned}
$$

Similarly, real functions g_{β_j} can be found by

$$
\begin{aligned}
&g_{\beta_j}(u) = \frac{1}{\beta_j!(2\pi)^{\beta_j}}\left(\sin^{\beta_j} 2\pi u - \sum_{l=1}^{\alpha_j - 1} a_l \sin^{\beta_j + l} 2\pi u\right),\\
&a_l = \frac{(2\pi)^{-\beta_j - l}}{(\beta_j + l)!}\frac{d^{\beta_j + l}}{du^{\beta_j + l}}\left(\sin^{\beta_j} 2\pi u - \sum_{r=1}^{l-1} a_r \sin^{\beta_j + r} 2\pi u\right)\Bigg|_{u=0}.
\end{aligned}
$$

Now it is not difficult to find a general form for polynomial masks $\tilde m_0$ providing wavelets with vanishing moments. Let $\tilde m_0$ satisfy condition (iii) of Theorem 2 with a set of parameters Λ_α. By (1),

$$
\tilde m_0\left(M^{*-1}x\right) - \tilde m_0^*\left(M^{*-1}x, \Lambda_\alpha\right) = \frac{1}{\sqrt{m}}\sum_{k=0}^{m-1} e^{2\pi i (r_k, x)}\left(\tilde\mu_k(x) - \tilde\mu_k^*(x)\right), \qquad (13)
$$

where μ_k^* is the kth polyphase representative of the function $\tilde m_0^*(\cdot, \Lambda_\alpha)$ defined by (10). There exist an algebraic polynomial P and $N \in \mathbb{Z}^d$ such that

$$
\tilde\mu_k(x) - \tilde\mu_k^*(x) = e^{2\pi i (N, x)} P(z), \qquad (14)
$$

where $z=(z_1,\dots,z_d)$, $z_j=e^{2\pi i(x,\mathbf{e}_j)}$. Since, due to Theorem 1,

$$D^\beta\big(\tilde\mu_k(x)-\tilde\mu_k^*(x)\big)\big|_{x=\mathbf{0}}=0,\quad \beta\in\mathbb{Z}_+^d,\ \beta\leqslant\alpha,$$

it is clear that P has vanishing derivatives up to order α at the point $(1,\dots,1)$. By the Taylor formula, there exist algebraic polynomials Q_j, $j=1,\dots,d$, so that

$$P(z)=\sum_{j=1}^d Q_j(z)(1-z_j)^{\alpha_j+1}.$$

Combining this with (13) and (14), we obtain

$$\tilde m_0(x)=\tilde m_0^*(x,\Lambda_\alpha)+\sum_{j=1}^d T_j(x)\big(1-e^{2\pi i(x,M\mathbf{e}_j)}\big)^{\alpha_j+1}, \tag{15}$$

where T_j are arbitrary trigonometric polynomials. Evidently, each function $\tilde m_0$ defined by (15) satisfies condition (iii) of Theorem 1 with a set of parameters Λ_α.

Similarly, all trigonometric polynomials $\tilde m_0$ satisfying condition (iii) of Theorem 5 with a set of parameters Λ^n can be described by

$$\tilde m_0(x)=\tilde m_0^*\big(x,\Lambda^n\big)+\sum_{[\alpha]=n+1} T_\alpha(z)\prod_{j=1}^d\big(1-e^{2\pi i(x,M\mathbf{e}_j)}\big)^{\alpha_j}, \tag{16}$$

where T_α are an arbitrary trigonometric polynomials.

Other general forms for polynomial masks providing vanishing moments can be given by

$$\tilde m_0(x)=\frac{1}{m}\sum_{k=0}^{m-1}e^{2\pi i(s_k,x)}\left(f_k(M^*x)-\sum_{0\leqslant\beta\leqslant\alpha}\big(D^\beta f_k(\mathbf{0})-\Delta_{\beta k}\big)g_\beta(M^*x)\right), \tag{17}$$

$$\tilde m_0(x)=\frac{1}{m}\sum_{k=0}^{m-1}e^{2\pi i(s_k,x)}\left(f_k(M^*x)-\sum_{0\leqslant[\beta]\leqslant n}\big(D^\beta f_k(\mathbf{0})-\Delta_{\beta k}\big)g_\beta(M^*x)\right), \tag{18}$$

where

$$\Delta_{\beta k}=\frac{1}{\sqrt{m}}\sum_{\mathbf{0}\leqslant\gamma\leqslant\beta}\lambda_\gamma\binom{\beta}{\gamma}(-2\pi i r_k)^{\beta-\gamma},$$

the functions f_k, g_β are trigonometric polynomials, $D^\gamma g_\beta(\mathbf{0})=\delta_{\beta\gamma}$ for all $\gamma\in\mathbb{Z}_+^d$, $\gamma\neq\beta$, $\mathbf{0}\leqslant\gamma\leqslant\alpha$ ($0\leqslant[\gamma]\leqslant n$ for (18)). If f_k, $k=0,\dots,m-1$, are arbitrary functions from $L_2(0,1)$ with continuous derivatives at the origin, then (17), (18) give a general form for all masks providing VM_α, VM^n properties with the sets of parameters Λ_α, Λ^n respectively. This follows from the equalities

$$D^\gamma\left(f_k(M^*x)-\sum_{0\leqslant\beta\leqslant\alpha}\big(D^\beta f_k(\mathbf{0})-\Delta_{\beta k}\big)g_\beta(M^*x)\right)\Bigg|_{x=\mathbf{0}}=\Delta_{\gamma k},\quad 0\leqslant\gamma\leqslant\alpha,$$

$$D^\gamma\left(f_k(M^*x)-\sum_{0\leqslant[\beta]\leqslant n}\big(D^\beta f_k(\mathbf{0})-\Delta_{\beta k}\big)g_\beta(M^*x)\right)\Bigg|_{x=\mathbf{0}}=\Delta_{\gamma k},\quad 0\leqslant[\gamma]\leqslant n.$$

4. Interpolatory masks

Finding a suitable refinable mask $\tilde m_0$ is the first step in construction of biorthogonal wavelets. After that, according to Unitary Extension Principle we should find a dual refinable mask m_0 and wavelet masks $m_\nu \tilde m_\nu$, $\nu = 1, \dots, m-1$, which is very complicated. This problem is closely related to the famous Serre conjecture stating that a unimodular line of algebraic polynomials can be extended to a unimodular matrix. The Serre conjecture was solved independently by Quillen and Suslin. Moreover, Suslin [10] proved an analog of this statement for a wider class of rings, in particular, for the ring of Laurent polynomials. On the basis of this Suslin's result, an appropriate first line of the matrix $\widetilde{\mathcal{M}}$ can be extended to a matrix whose entries are trigonometric polynomials and the determinant equals 1. After this it is not difficult to find a required matrices $\mathcal{M}, \widetilde{\mathcal{M}}$ (see [8,9]). Though the problem is solved theoretically, implementable algorithms for matrix extensions are not known in general. Next we will consider a class of refinable masks for which construction of wavelets may be realized in practice.

A mask $\tilde m_0$ is said to be *interpolatory* if $\tilde\mu_{00} \equiv \mathrm{const}$. Different methods for construction of some families of interpolatory masks providing vanishing moments were suggested in [11,12]. We will describe the class of all such masks.

Let $n \in \mathbb{Z}_+$, $\Lambda_0^n := \{\lambda_{\mathbf{0}} = 1, \lambda_\gamma = 0, \gamma \in \mathbb{Z}_+^d, [\gamma] \leqslant n, \gamma \ne \mathbf{0}\}$. For this set of parameters, the function $\tilde\mu_{00} \equiv \frac{1}{\sqrt m}$ evidently satisfies (9). If we choose $\tilde\mu_{0\nu}$, $\nu = 1, \dots, m-1$, so that

$$D^\beta \tilde\mu_{0k}(\mathbf{0}) = \frac{1}{\sqrt m}(-2\pi i r_k)^\beta, \quad \beta \in \mathbb{Z}_+^d,\ [\beta] \leqslant n, \tag{19}$$

then condition (iii) of Theorem 5 is valid. Inversely, for each interpolatory mask providing property VM^n, the set of parameters in (iii) is Λ_0^n. Using above arguments we obtain the following general form for polynomial interpolatory masks providing VM^n property:

$$\tilde m_0(x) = \tilde m_0^*\big(x, \Lambda_0^n\big) + \sum_{[\alpha]=n+1} T_\alpha(z) \prod_{j=1}^d \big(1 - e^{2\pi i (x, M\mathbf{e}_j)}\big)^{\alpha_j}, \tag{20}$$

where

$$\tilde m_0^*\big(x, \Lambda_0^n\big) = \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi i (s_k, x)} \sum_{\mathbf{0} \leqslant [\beta] \leqslant n} g_\beta(M^* x)(-2\pi i r_k)^\beta, \tag{21}$$

T_α are trigonometric polynomials with vanishing Fourier coefficients whose numbers are congruent to $\mathbf{0}$ modulo M.

A trivial dual mask of a given interpolatory mask $\tilde m_0$ is $m_0 \equiv m$. This mask does not provide vanishing moment. Numerical methods for finding dual masks providing vanishing moments (CBC algorithms, convolution method) were suggested in [13,14]. We consider a dual mask m_0 defined explicitly in polyphase form by

$$\mu_{00} = \sqrt m \left(1 - \sum_{k=1}^{m-1} |\tilde\mu_{0k}|^2\right), \quad \mu_{0k} = \tilde\mu_{0k},\ k = 1, \dots, m-1. \tag{22}$$

Such a mask m_0 is not interpolatory in general.

Proposition 6. *If an interpolatory refinable mask $\tilde{m}_0$ provides property VM^n, then its dual mask m_0 defined by* (22) *satisfies condition* (iii) *of Theorem* 5 *with the set of parameters Λ_0^n, i.e. m_0 also provides VM^n property.*

Proof. Since each function μ_{0k}, $k = 1, \dots, m-1$, coincides with $\tilde{\mu}_{0k}$, it satisfies (19). It remains to check that (19) is valid for μ_{00}. Evidently, $\mu_{00}(\mathbf{0}) = \frac{1}{\sqrt{m}}$. Let $\beta \in \mathbb{Z}_+^d$, $0 < [\beta] \leqslant n$. For any $k = 1, \dots, m-1$, using Leibniz formula and (19), we have

$$
\begin{aligned}
D^\beta \big|\tilde{\mu}_{0k}(x)\big|^2\big|_{x=0} &= D^\beta\big(\tilde{\mu}_{0k}(x)\overline{\tilde{\mu}_{0k}(x)}\big)\big|_{x=0} = \sum_{\mathbf{0}\leqslant\gamma\leqslant\beta}\binom{\beta}{\gamma} D^\gamma \tilde{\mu}_{0k}(\mathbf{0})\overline{D^{\beta-\gamma}\tilde{\mu}_{0k}(\mathbf{0})} \\
&= \frac{1}{m}\sum_{\mathbf{0}\leqslant\gamma\leqslant\beta}\binom{\beta}{\gamma}(-2\pi i r_k)^\gamma (2\pi i r_k)^{\beta-\gamma} = \frac{(2\pi i r_k)^\beta}{m}\sum_{\mathbf{0}\leqslant\gamma\leqslant\beta}\binom{\beta}{\gamma}\prod_{j=1}^d(-1)^{\gamma_j} \\
&= \frac{(2\pi i r_k)^\beta}{m}\prod_{j=1}^d \sum_{\mathbf{0}\leqslant\gamma_j\leqslant\beta_j}\binom{\beta}{\gamma}(-1)^{\gamma_j} = \frac{(2\pi i r_k)^\beta}{m}\prod_{j=1}^d(1-1)^{\beta_j} = 0.
\end{aligned}
$$

It follows that

$$
D^\beta \mu_{00}(\mathbf{0}) = 0, \quad \beta \in \mathbb{Z}_+^d,\ 0 < [\beta] \leqslant n.
$$

So, (19) holds true for $k = 0$. $\square$

Since extension of the line $\frac{1}{\sqrt{m}}, \tilde{\mu}_{01}, \dots, \tilde{\mu}_{0,m-1}$, to an unimodular matrix is trivial, using the method suggested in [9], we can easily find matrices $\mathcal{M}, \widetilde{\mathcal{M}}$:

$$
\mathcal{M} = \begin{pmatrix}
\sqrt{m}(1 - \sum_{k=1}^{m-1}|\tilde{\mu}_{0k}|^2) & \mu_{01} & \mu_{02} & \dots & \mu_{0,m-1} \\
-\overline{\mu_{01}} & 1/\sqrt{m} & 0 & \dots & 0 \\
-\overline{\mu_{02}} & 0 & 1/\sqrt{m} & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\overline{\mu_{0,m-1}} & 0 & 0 & \dots & 1/\sqrt{m}
\end{pmatrix}, \tag{23}
$$

$$
\widetilde{\mathcal{M}} = \begin{pmatrix}
\frac{1}{\sqrt{m}} & \mu_{01} & \mu_{02} & \dots & \mu_{0,m-1} \\
-\overline{\mu_{01}} & \sqrt{m}(1 - |\mu_{01}|^2) & -\sqrt{m}\,\overline{\mu_{01}}\mu_{02} & \dots & -\sqrt{m}\,\overline{\mu_{01}}\mu_{0,m-1} \\
-\overline{\mu_{02}} & -\sqrt{m}\,\overline{\mu_{02}}\mu_{01} & \sqrt{m}(1 - |\mu_{02}|^2) & \dots & -\sqrt{m}\,\overline{\mu_{02}}\mu_{0,m-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\overline{\mu_{0,m-1}} & -\sqrt{m}\,\overline{\mu_{0,m-1}}\mu_{01} & -\sqrt{m}\,\overline{\mu_{0,m-1}}\mu_{02} & \dots & \sqrt{m}(1 - |\mu_{0,m-1}|^2)
\end{pmatrix}. \tag{24}
$$

If it turned out that the dual refinable mask m_0, is also interpolatory, i.e.

$$
\frac{1}{m} + \sum_{k=1}^{m-1}|\mu_{0k}|^2 = 1,
$$

and hence the first lines of $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ coincide, then it is possible to extend this line to a unitary matrix $\mathcal{M} = \widetilde{\mathcal{M}}$ which can be found by means of Householder transform:

$$\mathcal{M} = \begin{pmatrix} \frac{1}{\sqrt{m}} & \mu_{01} & \mu_{02} & \cdots & \mu_{0,m-1} \\ \overline{\mu_{01}} & 1 - \frac{|\mu_{01}|^2}{1-1/\sqrt{m}} & \frac{-\overline{\mu_{01}}\mu_{02}}{1-1/\sqrt{m}} & \cdots & \frac{-\overline{\mu_{01}}\mu_{0,m-1}}{1-1/\sqrt{m}} \\ \overline{\mu_{02}} & \frac{-\overline{\mu_{02}}\mu_{01}}{1-1/\sqrt{m}} & 1 - \frac{|\mu_{02}|^2}{1-1/\sqrt{m}} & \cdots & \frac{-\overline{\mu_{02}}\mu_{0,m-1}}{1-1/\sqrt{m}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{\mu_{0,m-1}} & \frac{-\overline{\mu_{0,m-1}}\mu_{01}}{1-1/\sqrt{m}} & \frac{-\overline{\mu_{0,m-1}}\mu_{02}}{1-1/\sqrt{m}} & \cdots & 1 - \frac{|\mu_{0,m-1}|^2}{1-1/\sqrt{m}} \end{pmatrix}.$$

An orthogonal wavelet basis (or a tight frame) may be constructed in this way.

5. Example

Using general forms presented in Section 3, we can easily find a lot of concrete refinable masks providing vanishing moments. Due to the results of Section 4, we can also write wavelet masks for an interpolatory mask $\tilde{m}_0$ explicitly. However, only symmetric/antisymmetric wavelets are useful for some engineering application. So, we have two problems: first, we have to find an even refinable mask, and second, we need an appropriate matrix extension to get even/odd wavelet masks. For $d = 1$, the second problem was solved by Petukhov [15] for the orthogonal case $m_0 = \tilde{m}_0$. His algorithm is essentially not suitable for the multidimensional case. We will illustrate how it is possible to construct even or odd wavelet masks with vanishing moments for some matrix dilations with the help of above formulas.

Let $M = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$, for this matrix $m = 3$, $M^* = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$. If we choose $s_1 = \binom{1}{0}$, $s_2 = \binom{-1}{0}$, then $r_1 = \binom{1/3}{1/3}$, $r_2 = \binom{-1/3}{-1/3}$. By (21),

$$\tilde{m}_0^*\big(x, \Lambda_0^n\big) = \frac{1}{3}\left[1 + e^{2\pi i x_1} \sum_{0 \leqslant [\beta] \leqslant n} g_\beta(M^* x)(-2\pi i r_1)^\beta + e^{-2\pi i x_1} \sum_{0 \leqslant [\beta] \leqslant n} g_\beta(M^* x)(-2\pi i r_2)^\beta\right]$$

$$= \frac{1}{3}\left[1 + 2\cos 2\pi x_1 \sum_{\substack{0 \leqslant [\beta] \leqslant n \\ [\beta] \text{ is even}}} g_\beta(M^* x)\left(\frac{-2\pi i}{3}\right)^{[\beta]}\right.$$

$$\left. + 2i \sin 2\pi x_1 \sum_{\substack{0 \leqslant [\beta] \leqslant n \\ [\beta] \text{ is odd}}} g_\beta(M^* x)\left(\frac{-2\pi i}{3}\right)^{[\beta]}\right].$$

If we use real functions g_β, then $\tilde{m}^*$ is also real. Moreover, if we take an even function g_β whenever $[\beta]$ is even and an odd function g_β whenever $[\beta]$ is odd (this can be easily realized by the formula $\frac{1}{2}(g_\beta(x) + (-1)^{[\beta]} g_\beta(-x))$), then the mask $\tilde{m}^*$ is even.

It is not difficult to see that all the functions

$$\tilde{m}_0(x) = \tilde{m}_0^*\big(x, \Lambda_0^n\big) + \sum_{[\alpha]=n+1} T_\alpha(x) \prod_{j=1}^{2} \sin^{\alpha_j} 2\pi (M^* x, \mathbf{e}_j)$$

$$= \tilde{m}_0^*(x, \Lambda_0^n) + \sum_{[\alpha]=n+1} T_\alpha(x) \sin^{\alpha_1} 2\pi(2x_1 - x_2) \sin^{\alpha_2} 2\pi(x_1 + x_2),$$

where T_α is an even trigonometric polynomial whenever $[\alpha]$ is even and an odd trigonometric polynomial T_α whenever $[\alpha]$ is odd, are also even masks providing VM^n property.

For $n = 1$, choosing $g_0(u) = 1$, $g_1(u) = \frac{1}{2\pi}\sin 2\pi u$, we obtain the function

$$\tilde{m}_0(x) = \tilde{m}_0^*(x, \Lambda_0^1) = \frac{1}{9}\big[3 + 6\cos 2\pi x_1 + 4\sin 2\pi x_1 \sin 3\pi x_1 \cos\pi(x_1 - 2x_2)\big],$$

the table of its Fourier coefficients looks as follows

$$\begin{pmatrix} -1/18 & 0 & 1/18 & 1/18 & 0 & -1/18 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & -1/18 & 0 & 1/18 & 1/18 & 0 & -1/18 \end{pmatrix},$$

the polyphase representatives are

$$\begin{aligned}
\tilde{\mu}_{00}(x) &= \frac{1}{\sqrt{3}}, \\
\tilde{\mu}_{01}(x) &= \frac{1}{\sqrt{3}}\left(1 - \frac{i}{3}\sin 2\pi x_1 - \frac{i}{3}\sin 2\pi x_2\right), \\
\tilde{\mu}_{02}(x) &= \frac{1}{\sqrt{3}}\left(1 + \frac{i}{3}\sin 2\pi x_1 + \frac{i}{3}\sin 2\pi x_2\right).
\end{aligned}$$

Computing polyphase representatives of a dual mask m_0 by (22)

$$\begin{aligned}
\mu_{00}(x) &= \frac{1}{\sqrt{3}} - \frac{2}{9\sqrt{3}}(\sin 2\pi x_1 + \sin 2\pi x_2)^2, \\
\mu_{0k}(x) &= \tilde{\mu}_{0k}(x), \quad k = 1, 2,
\end{aligned}$$

we find

$$\begin{aligned}
m_0(x) = \frac{1}{27}\big(&9 + 18\cos 2\pi x_1 + 12\sin 2\pi x_1 \sin 3\pi x_1 \cos\pi(x_1 - 2x_2) \\
&- 8\sin^2 3\pi x_1 \cos^2\pi(x_1 - 2x_2)\big).
\end{aligned}$$

This function is also even. To find wavelet masks use the matrices (23), (24):

$$\begin{aligned}
\mathcal{M} &= \begin{pmatrix} \mu_{00} & \mu_{01} & \mu_{02} \\ -\overline{\mu_{01}} & 1/\sqrt{3} & 0 \\ -\overline{\mu_{02}} & 0 & 1/\sqrt{3} \end{pmatrix} =: \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \end{pmatrix}, \\
\widetilde{\mathcal{M}} &= \begin{pmatrix} \frac{1}{\sqrt{m}} & \mu_{01} & \mu_{02} \\ -\overline{\mu_{01}} & \sqrt{3}(1 - |\mu_{01}|^2) & -\sqrt{3}\,\overline{\mu_{01}}\mu_{02} \\ -\overline{\mu_{02}} & -\sqrt{3}\,\overline{\mu_{02}}\mu_{01} & \sqrt{3}(1 - |\mu_{02}|^2) \end{pmatrix} =: \begin{pmatrix} \tilde{Q}_0 \\ \tilde{Q}_1 \\ \tilde{Q}_2 \end{pmatrix}.
\end{aligned}$$

These two matrices are mutually inverse. It is clear that the matrices

$$\mathcal{M}' := \begin{pmatrix} Q_0 \\ \frac{1}{2}(Q_1 + Q_2) \\ \frac{1}{2i}(Q_1 - Q_2) \end{pmatrix}, \qquad \widetilde{\mathcal{M}}' := \begin{pmatrix} \tilde{Q}_0 \\ (\tilde{Q}_1 + \tilde{Q}_2) \\ \frac{1}{i}(\tilde{Q}_1 - \tilde{Q}_2) \end{pmatrix}$$

are also mutually inverse. Now, if we set

$$(\mu_{10}, \mu_{11}, \mu_{12}) := \frac{1}{2}(Q_1 + Q_2) = \left(-\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\right),$$
$$(\mu_{20}, \mu_{21}, \mu_{22}) := \frac{1}{2i}(Q_1 - Q_2) = \left(-\frac{\sin 2\pi x_1 + \sin 2\pi x_2}{3\sqrt{3}}, \frac{-i}{2\sqrt{3}}, \frac{i}{2\sqrt{3}}\right),$$

the corresponding wavelet masks are

$$m_1(x) = \frac{1}{3}(\cos 2\pi x_1 - 1),$$
$$m_2(x) = \frac{1}{9}\big(3\sin 2\pi x_1 - 2\sin 3\pi x_1 \cos \pi(x_1 - 2x_2)\big).$$

For $n = 2$, choosing $g_0(u) = 1$, $g_1(u) = \frac{1}{2\pi}\sin 2\pi u$, $g_2(u) = \frac{1}{8\pi^2}\sin^2 2\pi u$, we obtain a refinable mask

$$\tilde{m}_0(x) = \tilde{m}_0^*\big(x, \Lambda_0^2\big) = \frac{1}{27}\big[9 + 18\cos 2\pi x_1 + 12\sin 2\pi x_1 \sin 3\pi x_1 \cos \pi(x_1 - 2x_2)$$
$$- 4\cos 2\pi x_1 \sin^2 3\pi x_1 \cos^2 \pi(x_1 - 2x_2)\big],$$

the table of its Fourier coefficients looks as follows

$$\begin{pmatrix} \frac{1}{216} & 0 & \frac{1}{216} & -\frac{1}{108} & 0 & -\frac{1}{108} & 0 & 0 & \frac{1}{216} & 0 & 0 \\ 0 & 0 & -\frac{1}{18} & 0 & \frac{1}{18} & \frac{1}{18} & 0 & -\frac{1}{18} & 0 & 0 & 0 \\ 0 & \frac{1}{108} & 0 & \frac{1}{108} & \frac{17}{54} & \frac{1}{3} & \frac{17}{54} & \frac{1}{108} & 0 & \frac{1}{108} & 0 \\ 0 & 0 & 0 & -\frac{1}{18} & 0 & \frac{1}{18} & \frac{1}{18} & 0 & -\frac{1}{18} & 0 & 0 \\ 0 & 0 & \frac{1}{216} & 0 & \frac{1}{216} & -\frac{1}{108} & 0 & -\frac{1}{108} & \frac{1}{216} & 0 & \frac{1}{216} \end{pmatrix}.$$

Similarly to the case $n = 1$, we can find the following dual refinable mask m_0 and wavelet masks m_1, m_2:

$$m_0(x) = \frac{1}{243}\big(81 + 162\cos 2\pi x_1 + 108\sin 2\pi x_1 \sigma(x) - 36\cos 2\pi x_1 \sigma^2(x) - 8\sigma^4(x)\big),$$
$$m_1(x) = \frac{1}{81}\big(-9 + 27\cos 2\pi x_1 + 2\sigma^2(x)\big),$$
$$m_2(x) = \frac{1}{9}\big(-3\sin 2\pi x_1 + 2\sigma(x)\big),$$

where $\sigma(x) = \sin 3\pi x_1 \cos \pi(x_1 - 2x_2)$.

By construction, we have

$$\sum_{k=0}^{m-1} \overline{\tilde{\mu}_{0k}} \mu_{0k} = 1.$$

This is a necessary condition for biorthogonality of the scaling functions φ, $\tilde{\varphi}$ with the masks m_0, $\tilde{m}_0$. To prove a sufficient condition we will use Cohen criterion (which was extended to the multivariate biorthogonal case with an arbitrary matrix dilation in [16]). The set $P := M^{*-1}([-1/2, 1/2]^2)$ is the parallelogram depicted on Fig. 1a. By cutting and removing the triangles $\Delta_1 := P \cap \{x \in \mathbb{R}^2\colon x_1 \geqslant 1/5\}$, $\Delta_2 := P \cap \{x \in \mathbb{R}^2\colon x_1 \leqslant -1/5\}$ as it is depicted on Fig. 1b, we obtain a polygon Ω. It is clear that the set $K := M^*\Omega$ is a Cohen compact. Since $\cos 2\pi x_1 > 0.3$ whenever $|x_1| \leqslant 1/5$, we have

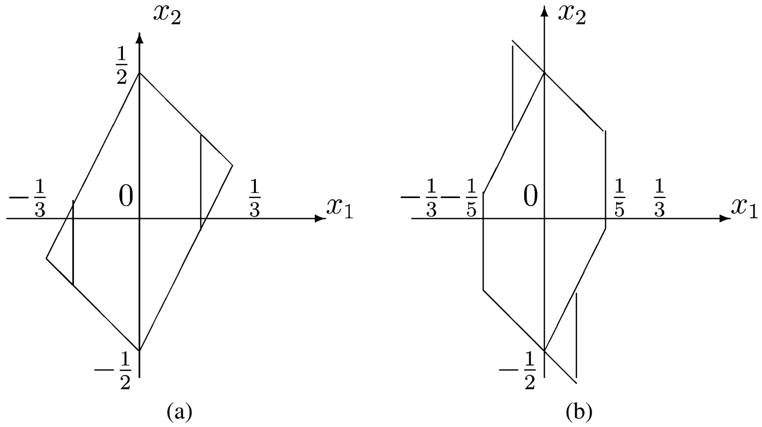

Fig. 1. (a) Parallelogram P; (b) Polygon Ω.

$$\tilde{m}_0(x) \geqslant \frac{1}{27}\big(9 + \cos 2\pi x_1\big(18 - 4\sigma^2\big) + 12 \sin 2\pi x_1\sigma\big) > (9 + 4 - 12) > 0,$$

$$m_0(x) \geqslant \frac{1}{243}\big(81 + \cos 2\pi x_1\big(162 - 36\sigma^2\big) + 108 \sin 2\pi x_1\sigma - 8\sigma^4\big) > (81 + 37 - 108 - 8) > 0$$

for all $x \in \{x \in \mathbb{R}^2\colon\ |x_1| \leqslant 1/5\} =: S$. We proved that $\Omega \subset S$. It is not difficult to see that $M^{*-k}\Omega \subset S$ for $k = 1, 2, \ldots$. Hence,

$$\inf_{k\in\mathbb{N}} \inf_{x\in K} \big|m_0\big(M^{*-k}x\big)\big| \neq 0, \qquad \inf_{k\in\mathbb{N}} \inf_{x\in K} \big|\tilde{m}_0\big(M^{*-k}x\big)\big| \neq 0,$$

which means that Cohen criterion is fulfilled.

The same method for finding refinable and wavelet masks with symmetric properties may be used for a wide class of matrices with an odd determinant.

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